Modeling population dynamics with random initial conditions by means of statistical moments

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Abstract

In this paper a random differential equation system modeling population dynamics is investigated by means of the statistical moments equation. Monte Carlo simulations are performed in order to compare with the statistical moments equation approach. The randomness in the model appears due to the uncertainty on the initial conditions. The model is a nonlinear differential equation system with random initial conditions and is based on the classical SIS epidemic model. By assuming different probability distribution functions for the initial conditions of different classes of the population we obtain the mean and variance of the stochastic process representing the proportion of these classes at any time. The results show that the theoretical approach of moments equation agrees very well with Monte Carlo numerical results and the solutions converge to the equilibrium point independently of the probability distribution function of the initial conditions.

Keywords: Random differential equation, Statistical moments equation, Stochastic process, Population dynamics, SIS epidemic model, Monte Carlo method.

AMS Subject classifications: 37N25, 65C05, 81T80, 91D10, 91B74, 93A30.
1. Introduction

Mathematical models dealing with uncertainty in differential equations have been considered in the recent decades in a wide variety of applied areas, such as physics, chemistry, biology, economics, sociology and medicine. In many situations, equations with random inputs are better suited in describing the real behavior of quantities of interest than their counterpart deterministic equations. Randomness in the input may arise because of errors in the observed or measured data, variability in experiment and empirical conditions, uncertainties (variables that cannot be measured or missing data) or plainly because of lack of knowledge as mentioned in Chen-Charpentier et al. [2] and Möller et al. [12]. When data are available to inform the choice of distribution, the parameter assignment is easily made. However, in the absence of data to inform on the distribution for a given parameter, it is usually assumed Uniform, Gaussian and Beta distributions, such as in the papers of Kegan and West [9]; Ju [8]; Korsunskii et al. [10] and Gupta [5]. It is important to mention that a previous paper regarding random differential equations for a susceptible-infected (SI) epidemic model has been developed in Kegan and West [9], where the only transition in the individual of the population are from susceptible to infected. In addition, in this previous paper birth and death process are not considered. In this paper the authors consider uncertainty into the initial conditions using the Beta probability density function. They also compute a probability distribution that describes the mean and variance of the proportion of susceptible at any time during an epidemic. Nevertheless, in our work the random differential equation model is more complex since it includes other transitions in the whole population.

In this work, our main aim is to investigate a random differential equation system modeling the evolution of the 24-65 years old excess weight populations of the region of Valencia (Spain) in order to study the effect that population initial condition uncertainty has on the dynamics of the population under study. Other interesting models treating obesity dynamics considering both individual and population groups have been presented in Navarro-Barrientos et al. [13]; Santonja et al. [16]; González-Parra et al. [4] and Jódar et al. [7]. However, in these previous works initial conditions have been assumed deterministic. Here, we assume that population initial conditions of the model follow well-known distributions such the Gaussian, Uniform and Beta where their parameters are computed in order to obtain the same mean value of the sample data of the real populations of the Region of Valencia (Spain). It is important to point out that the analysis of uncertainty in other parameters or for a system with more equations is not feasible by means of the statistical moments equations due to the increased complexity. Therefore, in this article uncertainty is considered only on the initial conditions. However, Monte Carlo method allows much more complexity in the random differential equation system.
Besides the aforementioned aim we develop Monte Carlo simulations in order to compare moments equation and numerical results assuming different probability distribution functions on the initial conditions. The theoretical results are derived by means of the statistical moments equation presented in Soong [17]. In this paper we obtain theoretical results regarding the mean and variance of the proportion of normal and excess weight populations at any time during the social epidemic. The approach of using the moments equation has been applied successfully to approximate the first and second order moments of the stochastic Hodgkin–Huxley system describing spiking neurons in Tuckwell and Jost [18].

The versatility of Monte Carlo simulation modeling allows us to include more complexity into the deterministic mathematical models. Following this way the Monte Carlo method is a powerful method for assessing the impact of uncertainties due to the model inputs such as performed in Mallet and Sportisse [11]. Hence, random effects can be included using Monte Carlo simulations and using different probability distribution functions for the initial conditions. The Monte Carlo method has been used successfully in several works of different areas such in Xiu and Karniadakis [21] and Hanna et al. [6]. For instance, in Rasulov et al. [15] Monte Carlo method has been used to solve the Cauchy problem for a non-linear parabolic equation. Additionally, Monte Carlo is a classical method to integrate and has been used in several works such in Pillards et al. [14].

Monte Carlo method described in Fishman [3] is the classical and most used technique for approximating expected values of quantities of interest depending on the solution of differential equations with random inputs. The algorithm approximates the desired expectation by a sample average of independent identically distributed (iid) realizations. When solving differential equations with random inputs, this method implies the solution of one deterministic differential equation for each realization of the input parameters. This makes the method simple to implement, maximum code reusability and it is straightforward to parallelize. Its numerical error has order $O(1/\sqrt{M})$, where $M$ is the number of realizations. The advantage of using this approach is that the above rate error does not deteriorate with respect to the number of random variables in the problem, making the method very attractive for problems with large dimensional random inputs. Based on the aforementioned information, Monte Carlo method is used here with the aim of comparing the moments equation and the numerical results assuming different probability distribution functions on the initial conditions. Comparison between moments equation and the numerical results verify clearly the consistency between the moments equation and Monte Carlo method. Thus, approximate mean and variance of the process solution for the different populations can be obtained through Monte Carlo simulations in order to predict the possible dynamics of excess weight population over the time horizon.

It is worth to point out here the difficulties to obtain confident data and the
importance of introducing randomness at least on the initial conditions. For instance, in the Spanish region of Valencia, a health survey is done every 5 years and data should be prepared, processed and stored in databases before their availability. Moreover surveys are exposed to human errors and their costs are very high. Initial conditions of the mathematical model are the prevalence of excess weight in the population of the region of Valencia (Spain) corresponding to the year 2000.

The paper is organized as follows. In Section 2 the deterministic mathematical model of obesity population is presented. Section 3 deals with the computation of the stochastic process solution. In Section 4 numerical results are computed using the statistical moments equation and Monte Carlo simulations. Finally, Section 5 is devoted to a short discussion and conclusions.

2. The mathematical model

In this paper a mathematical model for the evolution of excess weight population under uncertainty is investigated. This model is based on the partition of the adult population into two subpopulations. In this model \( N(t) \) denotes the proportion of normal weight individuals and \( O(t) \) the proportion of excess weight individuals. Without loss of generality and for the sake of clarity, 24-65 years old adult population is normalized to unity, and one gets for all time \( t \),

\[ N(t) + O(t) = 1. \]

The model is represented by the following nonlinear system of ordinary differential equations:

\[
\begin{align*}
N'(t) &= \mu N_0 - \mu N(t) - \beta N(t)O(t) + \rho O(t), \\
O'(t) &= \mu O_0 + \beta N(t)O(t) - (\mu + \rho)O(t),
\end{align*}
\]

(2.1)

with random initial conditions \( N_1 \) and \( O_1 \), such that \( N(0) = N_1 \) and \( O(0) = O_1 \). The time invariant parameters of the system (2.1) are:

- \( \mu \), is inversely proportional to the mean time spent on the system for 24–65 years old adults.
- \( \rho \), rate at which a excess weight individual moves to the normal weight subpopulation.
- \( \beta \), transmission rate due to social pressure to adopt a unhealthy lifestyle.
- \( N_0 \), proportion of normal weight coming from the 23 years old age group.
- \( O_0 \), proportion of excess weight coming from the 23 years old age group.
Throughout this paper, we focus on the dynamics of the model \((2.1)\) in the following restricted region:
\[
\sigma = \{ (N, O) / N > 0, O > 0, \ N + O = 1 \},
\]
where the basic results as usual local existence, uniqueness and continuation of solutions are valid for system \((2.1)\). This nonlinear model with the initial conditions and parameter values shown in Table 1 has an equilibrium point at \(N^* = 0.34\) and \(O^* = 0.66\). Parameter \(\beta\) is estimated by fitting the model \((2.1)\) to the available data of the Health Survey of the Region of Valencia 2000 and 2005 available in Valencian Health Surveys [19] and Valencian Health Surveys [20]. In particular the initial conditions \((N(0), O(0))\) together with final conditions \((N(260), O(260))\) are used as the fitting points. The values of \(N_0\) and \(O_0\) correspond to the proportion of normal and excess weight individuals in the 23 years old age group for year 2000. The other parameter \(\rho\) is estimated taking into account the mean time that an individual takes after he/she stops physical activity to start again. These data is taken from Valencian Health Surveys [19, 20] and Arrizabalaga et al. [1]. The dynamics of transits between subpopulations is depicted graphically in Figure 1.

<table>
<thead>
<tr>
<th>(N(0))</th>
<th>(O(0))</th>
<th>(\beta)</th>
<th>(\rho)</th>
<th>(\mu)</th>
<th>(N_0)</th>
<th>(O_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.522</td>
<td>0.488</td>
<td>0.00008</td>
<td>0.000035</td>
<td>0.00046</td>
<td>0.704</td>
<td>0.296</td>
</tr>
</tbody>
</table>

Table 1: Initial conditions and parameter values for the SIS model.

![Figure 1](image)

Figure 1: Flow diagram of the mathematical model for the dynamics of obesity prevalence in the population.

3. Computing the solution process

In order to compute the solution process of random system \((2.1)\) and for the sake of clarity, the compartmental random mathematical model \((2.1)\) can be simplified to one differential equation and represented analytically by the following random nonlinear ordinary differential equation:
\[
N' (t) = A + B N (t) + C (N(t))^2, \quad N (0) = N_1,
\] (3.1)
where $A = \mu N_0 + \rho$, $B = -(\rho + \mu + \beta)$, $C = \beta$ and $N_1$ is a random variable representing the initial normal weight population. In addition, the unknown $N(t)$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. A solution $N(t)$ of random differential equation (3.1) means that for each $\omega \in \Omega$, $N(t)(\omega), \omega \in \Omega$, satisfies the deterministic problem obtained from (3.1) taking realizations of the involved random variable and where derivatives and limits are regarded in the mean square sense, see Soong [17] for details. Since the component of the solution is a stochastic process, we can rely on Monte Carlo method to compute the expected solution using in particular the forward Euler method with a small step size $\Delta t$ and coded with Matlab software.

The counterpart deterministic ordinary differential equation of (3.1) can be solved analytically and the solution is given by the following expression:

$$N(t) = \frac{B - \tan \left( \frac{t \sqrt{4AC - B^2}}{2} + \arctan \left( \frac{2C N_1 + B}{\sqrt{4AC - B^2}} \right) \right)}{2C}.$$ (3.2)

Since the random variable $N_1$ is a proportion, its support is the interval $[0, 1]$. Some choices for its distribution include Uniform, truncated Gaussian and Beta distributions, with their respective parameters. It is important to mention that the Beta probability distribution $B(\alpha, \beta)$ includes the Uniform $[0, 1]$ distribution when the parameters $\alpha = 1$ and $\beta = 1$.

There are different methods for determining the joint density function of the mean square solution $N(t)$, as mentioned in Soong [17]. In the case that randomness enters into the model only through the initial condition, this determination presents no conceptual difficulty if the solution of the corresponding deterministic differential equation (3.1) is found, as mentioned in Soong [17], Ch. 6. Thus, a great information about the statistical behavior of its mean square solution can be usually obtained. However, the statistical moments such as mean and variance of the solution process can be found in a simpler way. They can be determined directly from the explicit form of the solution process, see Soong [17], Ch. 6 for details. Hence, the $n$-th moment of $N(t)$, is given by

$$E[N(t)^n] = \int_{-\infty}^{\infty} (h(N1, t))^n f_0(N1) dN1,$$ (3.3)

where $h(N1, t)$ is the solution process of random differential equation (3.1) and $f_0(N1)$ is the density function of the random initial condition $N1$.

As it can be seen in equation (3.2) the solution process expression depends on trigonometric functions in a quite complex way. Therefore, despite the simple form of relation (3.3), in order to obtain the statistical moments we have to rely on numerical integration since a closed form is not easily obtained. The variance
can be computed using relation (3.3) and the well-known expression:

\[ V[N(t)] = E[N(t)^2] - (E[N(t)])^2, \]

where \( E[\cdot] \) denotes the expectation operator.

It is important to remark that when explicit solutions are not available it is necessary to rely on numerical methods to approximate the solution of the ordinary differential equation system and the moments equation. Next section is devoted to compute the mean and variance of the solution process by means of expressions (3.2)-(3.4). The probability density functions included in this computation are the well-known truncated Gaussian, Uniform and Beta distributions. Using the relationship \( N(t) + O(t) = 1 \) between the proportion of normal and excess weight population we can obtain the proportion of excess weight population at any time \( t \), once the proportion of normal weight population is known.

4. Numerical results for the mean and standard deviation of the solution process

As we have pointed out in the previous section, now we address the computation of the mean and variance of the solution process by means of relations (3.2)-(3.4). Monte Carlo simulations are also included in order to support the moments equation results and verify clearly the numerical agreement of this theoretical approach and Monte Carlo method. With Monte Carlo method the mathematical model needs to be simulated to obtain output results using the probability density function prescribed for the initial condition. The process is repeated many times in order to obtain large amount of data.

As it has been mentioned in the introduction when data are available to inform the choice of distribution, the parameter assignment is easily made. However, in the absence of data to inform on the distribution for a given parameter, it is usually assumed Uniform, Gaussian and Beta distributions, such as in the papers of Kegan and West [9]; Ju [8] and Gupta [5]. In probability theory and statistics, the beta distribution is a family of continuous probability distributions defined on the interval \([0, 1]\) parameterized by two positive shape parameters, typically denoted by \( \alpha \) and \( \beta \).

The truncated Gaussian distribution has the same mean that the classical Gaussian distribution if the lower and upper limits of the support are symmetric respect to the original mean. The Beta density function can take on different shapes depending on the values of the two parameters. For instance, if \( \alpha = 1 \) and \( \beta = 1 \) one gets the Uniform \([0, 1]\) distribution. In addition, the mean is given by \( \frac{\alpha}{\alpha + \beta} \) and the variance by \( \frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \). Moreover, if \( \alpha = \beta \) then the density function is symmetric about 0.5.
4.1. Mean of the solution process

At first we compute the mean of the solution process by means of expression (3.3) for \( n = 1 \) using Maple software which uses a hybrid symbolic-numeric solution strategy that include Clenshaw-Curtis quadrature. In addition, we compute the approximate mean of the solution process by using Monte Carlo method and forward Euler scheme. Figure 2 shows the expectation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a Uniform probability density function with support on the interval \([0.422, 0.622]\) by means of the moments equation (3.3) and Monte Carlo method. The interval has been chosen so that the mean of the distribution is equal to the computed initial condition coming from the real data and assuming that this value may have an error \( \pm 0.1 \). However, other intervals have been used and the results do not vary qualitatively. Numerical results show that the moments equation and Monte Carlo results agree very well. As expected by the law of large numbers the Monte Carlo method increases its accuracy as the number of realizations increases. On the other hand, evolution of the mean value of the proportion of normal weight population for different weeks in the model (2.1) computed by the statistical moments equation is shown in Table 2. Notice that the mean value converges to the equilibrium point \( N^* = 0.34 \).

<table>
<thead>
<tr>
<th>Weeks</th>
<th>( E[N(t)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.522</td>
</tr>
<tr>
<td>1000</td>
<td>0.44</td>
</tr>
<tr>
<td>2000</td>
<td>0.39</td>
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<tr>
<td>3000</td>
<td>0.36</td>
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<tr>
<td>4000</td>
<td>0.35</td>
</tr>
<tr>
<td>5000</td>
<td>0.34</td>
</tr>
</tbody>
</table>

Table 2: Evolution of the mean value of the proportion of normal weight population for different weeks in the proposed model (2.1) computed by the statistical moments equation for the Uniform case.

Figure 3 shows the expectation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a truncated Gaussian probability density function with parameters \((\mu = 0.522, \sigma = 0.05)\) on the interval \([0.222, 0.822]\) in order to avoid unreal initial conditions. The parameter value for \( \sigma \) is chosen in order to obtain approximately the same standard deviation of the Uniform distribution and to make balanced numerical comparisons. As in the previous case the moments equation and Monte Carlo results agree very well as the number of realizations increases. In both cases it can be observed that the solution process corresponding to the proportion of normal weight population converges asymptotically to the equilibrium point. In addition, numerical results show that the initial condition does not affect the
steady state.

The case when the initial condition is taken from a Beta distribution $B(\alpha = 2, \beta = 2)$ is shown in Figure 4, where it can be seen that both results agree very well. The parameter values of the Beta distribution are chosen in order to obtain a bigger variance than the Uniform and the truncated Gaussian distributions corresponding to the population initial conditions. However, as it can be seen in Figure 4 this larger dispersion does not change the qualitative behavior of the solution process. Notice that in all cases the results present good agreement despite using different probability distributions functions.

![Figure 2: Expectation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a Uniform probability density function on the interval $[0.422, 0.622]$ by means of the moments equation (3.3) and Monte Carlo method with $m = 50$ and $m = 500$ realizations.](image)

**4.2. Standard deviation of the solution process**

Here we compute the standard deviation of the solution process by means of expressions (3.2), (3.3) and (3.4) with Maple software. In addition, we compute the approximate standard deviation of the solution process by using Monte Carlo method and forward Euler scheme. The computation of the standard deviation is from a computational point of view more expensive than the mean due to the fact that we need to compute numerically expression (3.3) with $n = 2$, instead of $n = 1$. Therefore, in the computation of the standard deviation of the solution process when population initial condition is assumed to follow a truncated Gaussian and Beta distributions we reduce the simulation time in order to avoid large computational times.

In Figure 5 it can be seen the standard deviation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a Uniform probability density function on the interval $[0.422, 0.622]$. 
Population dynamics with random initial conditions

Figure 3: Expectation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a truncated Gaussian probability density function with parameters $\mu = 0.522$ and $\sigma = 0.05$, on the interval $[0.222,0.822]$ by means of the moments equation (3.3) and Monte Carlo method with $m = 50$ and $m = 500$ realizations.

Figure 4: Expectation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a Beta probability density function $B(\alpha = 2, \beta = 2)$ by means of the moments equation (3.3) and Monte Carlo method with $m = 50$ and $m = 500$ realizations.
Numerical results show that Monte Carlo method increases its accuracy as the number of realizations increases. Table 3 shows the evolution of the standard deviation of the proportion of normal weight population for different weeks in the model (2.1) computed by the statistical moments equation.

<table>
<thead>
<tr>
<th>Weeks</th>
<th>Standard deviation of $[N(t)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.057</td>
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<tr>
<td>1000</td>
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<td>2000</td>
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<td>4000</td>
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</tr>
<tr>
<td>5000</td>
<td>0.0027</td>
</tr>
</tbody>
</table>

Table 3: Evolution of the standard deviation of the proportion of normal weight population for different weeks in the model (2.1) computed by the statistical moments equation for the Uniform case.

On the other hand, Figure 6 shows the standard deviation of the solution process when the initial condition is assumed to follow a truncated Gaussian probability density function with parameters $\mu = 0.522$ and $\sigma = 0.05$ on the interval $[0.222, 0.822]$. As in the previous case, Monte Carlo method performs very well as the number of realizations increases.

Finally in Figure 7 it can be seen the standard deviation of the solution process for the proportion of normal weight population when the initial condition is assumed that follows a Beta probability density function $B(\alpha = 2, \beta = 2)$. Numerical results show Monte Carlo method reliability and versatility.

5. Conclusions

In this paper a mathematical model based in the classical SIS epidemic model was presented in order to predict the evolution of excess weight adult population subject to uncertainty on the initial conditions. The model is represented by a system of nonlinear differential equations with random initial conditions. By assuming different probability distribution functions for the initial conditions of the normal and excess weight populations we obtain the mean and variance of the stochastic process representing the proportion of normal and excess weight populations at any time. The mean and variance for each chosen probability distribution functions are derived by means of the statistical moments equation and Monte Carlo simulations. The moments equation and Monte Carlo numerical results agree very well. In fact numerical results show that the Monte Carlo method increases its accuracy as the number of realizations increases as was expected. The comparisons verify clearly the consistency of the theoretical approach and the Monte Carlo method as a consequence of the law of large numbers.
Population dynamics with random initial conditions

Figure 5: Standard deviation of the solution process for the proportion of normal weight population when the initial condition is assumed to follow a Uniform probability density function on the interval \([0.422, 0.622]\) by means of the moments equation (3.3) and Monte Carlo method with \(m = 50\) and \(m = 500\) realizations.

Figure 6: Standard deviation of the solution process for the proportion of normal weight population when the initial condition is assumed to follow a truncated Gaussian probability density function with parameters \(\mu = 0.522\) and \(\sigma = 0.05\), on the interval \([0.222, 0.822]\) by means of the moments equation (3.3) and Monte Carlo method with \(m = 50\) and \(m = 500\) realizations.
Randomness on the initial conditions is justified since population data should be prepared, processed and stored in databases before their availability and these processes are exposed to human errors. It was found that the mean of the solution process corresponding to the proportion of normal weight converges to the equilibrium point for several combinations of the values of the parameters corresponding to each chosen probability distribution function. On the other hand the variance of the solution process corresponding to the proportion of normal weight was found to decrease as time increases since the solution process converges to the equilibrium point.

In this paper only the initial proportion of normal and excess weight populations was assumed to be random. Future models may benefit by the consideration of an initial distribution on the transmission rate, which is also likely not be known with certainty for a given population. The approach used here may be generalized to more complex models with more equations. However, more numerical algorithms would need to be used because closed form solutions would not exist. Finally it should be mentioned that the approach of this paper focuses on the dynamics of excess weight population, but could be extended to other types of epidemic modeling in future research.
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