A simple proof of Fisher’s Theorem and of the distribution of the sample variance statistic

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Abstract
In this paper a very simple and short proofs of Fisher’s theorem and of the distribution of the sample variance statistic in a normal population are given.

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1. Introduction
Let \( X = (X_1, \ldots, X_n) \) be a random vector such that \( E[X_i] = \mu \) and \( \text{Cov}[X_i, X_j] = \sigma^2 \delta_{ij} \) with \( \sigma^2 > 0 \) and \( \delta_{ij} = 1 \) if \( i = j \) and 0 otherwise. For \( n \geq 2 \), the mean of order \( k \) (\( 1 \leq k \leq n \)), defined as \( \bar{X}_k = \frac{1}{k} \sum_{i=1}^{k} X_i \), is considered, which verifies:

I. \( E[\bar{X}_k] = \mu \), since \( E[\bar{X}_k] = E\left[ \frac{1}{k} \sum_{i=1}^{k} X_i \right] = \frac{1}{k} \sum_{i=1}^{k} E[X_i] = \mu. \)

II. \( \text{Cov}[\bar{X}_\ell, \bar{X}_k] = \sigma^2 k \) for any \( 1 \leq \ell \leq k \leq n \) because

- If \( \ell = k \) then \( \text{Cov}[\bar{X}_\ell, \bar{X}_k] = \text{Var}[\bar{X}_k] = \frac{1}{k^2} \sum_{i=1}^{k} \text{Var}[X_i] = \frac{\sigma^2}{k}. \)
- If \( \ell < k \) then \( \text{Cov}[\bar{X}_\ell, \bar{X}_k] = \frac{1}{k} \text{Cov}[\bar{X}_\ell, X_1 + \ldots + X_\ell + \ldots + X_k] = \frac{1}{k} \text{Cov}[\bar{X}_\ell, \ell \bar{X}_\ell] = \frac{\ell}{k} \text{Var}[\bar{X}_\ell] = \frac{\sigma^2}{k}. \)

Let \( Y = (Y_2, \ldots, Y_n) \) be a random vector where

\[
Y_k = \frac{\sqrt{k(k-1)}}{\sigma} (\bar{X}_k - \bar{X}_{k-1}) \tag{1.1}
\]

for any \( 2 \leq k \leq n \). The most relevant properties of \( Y \) are:
III. \( E[Y_k] = 0 \) since \( E[\bar{X}_k - \bar{X}_{k-1}] = \mu - \mu = 0 \) from (I).

IV. \( \text{Cov}[Y_{\ell}, Y_k] = \delta_{\ell k}, \) since from (II),

- If \( \ell = k \) then \( \text{Var}[\bar{X}_k - \bar{X}_{k-1}] = \text{Var}[\bar{X}_k] = 2 \text{Cov}[\bar{X}_k, \bar{X}_{k-1}] + \text{Var}[\bar{X}_{k-1}] = \sigma^2 \left( \frac{1}{k} - \frac{2}{k^2} + \frac{1}{k^3} \right) = \frac{\sigma^2}{k(k-1)}, \) which implies that \( \text{Var}[Y_k] = 1 \)
- For \( 2 \leq \ell < k \leq n, \) then \( \text{Cov}[\bar{X}_\ell - \bar{X}_{\ell-1}, \bar{X}_k - \bar{X}_{k-1}] = \text{Cov}[\bar{X}_\ell, \bar{X}_k] - \text{Cov}[\bar{X}_\ell, \bar{X}_{k-1}] - \text{Cov}[\bar{X}_{\ell-1}, \bar{X}_k] + \text{Cov}[\bar{X}_{\ell-1}, \bar{X}_{k-1}] = \sigma^2 \left( \frac{1}{k} - \frac{1}{k^2} + \frac{1}{k^3} \right) = 0, \) which implies that \( \text{Cov}[Y_{\ell}, Y_k] = 0. \)

V. \( \text{Cov}[Y_{\ell}, \bar{X}_k] = 0 \) for \( 2 \leq \ell \leq k \leq n \) as:
\[ \text{Cov}[\bar{X}_\ell, \bar{X}_k] = \text{Cov}[\bar{X}_\ell, \bar{X}_{k-1}] = \sigma^2 - \frac{\sigma^2}{\ell} = 0. \]

VI. The statement \( \sigma^2 \sum_{k=2}^{n} Y_k^2 = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \) holds for \( n \geq 2. \) This equality is proven by induction on \( n. \)

For \( n = 2: \) \( \bar{X}_2 = \frac{1}{2}(X_1 + X_2) \) and as \( \bar{X}_1 = X_1, \) it follows that:
\[ (\bar{X}_2 - \bar{X}_1)^2 + (\bar{X}_2 - X_1)^2 = 2(\bar{X}_2 - X_1)^2 = 2(\bar{X}_2 - \bar{X}_1)^2 = \sigma^2 Y_2^2. \]
Assume the equality holds for \( n. \)
\[ \sum_{i=1}^{n+1} (X_i - \bar{X}_{n+1})^2 = \sum_{i=1}^{n} (X_i - \bar{X}_{n+1})^2 + (X_{n+1} - \bar{X}_{n+1})^2 = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 + n(\bar{X}_n - \bar{X}_{n+1})^2 + (X_{n+1} - \bar{X}_{n+1})^2. \]
But \( X_{n+1} = (n+1)\bar{X}_{n+1} - n\bar{X}_n \) and therefore \( (X_{n+1} - \bar{X}_{n+1})^2 = n^2(\bar{X}_{n+1} - \bar{X}_n)^2. \) Hence \( \sum_{i=1}^{n+1} (X_i - \bar{X}_{n+1})^2 = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 + (n^2 + n)(\bar{X}_{n+1} - \bar{X}_n)^2 \) = (applying the induction hypothesis) \( = \sigma^2 \sum_{k=2}^{n} Y_k^2 + \sigma^2 Y_{n+1}^2 = \sigma^2 \sum_{k=2}^{n+1} Y_k^2, \) and the equality holds for \( n+1. \)

From these results, it follows that:

**Proposition 1.1.** If the population model \( X \) is normal, then \( nS^2/\sigma^2 \sim \chi^2_{n-1}, \)
where \( S^2 \) is the sample variance statistic of \( X \) and \( \chi^2_{n-1} \) denotes the chi-squared distribution with \( n-1 \) degrees of freedom.

**Proof.** \( Y = (Y_2, \cdots, Y_n) \) with \( Y_k \) defined in (1.1) is normal since it is obtained from \( X \) by a linear transformation. Furthermore, from (VI): \( nS^2/\sigma^2 = \sum_{k=2}^{n} Y_k^2 \sim \chi^2_{n-1} \)
because the \( Y_k \) random variables are normal, their mean is zero (III), their variance is one (IV), and they are independent since they are uncorrelated (IV).

**Theorem 1.1.** (Theorem of Fisher). The statistics \( \bar{X} \) and \( S^2 \) are independent if the population model is normal.

**Proof.** The \( (Y_2, \cdots, Y_n, \bar{X}_n) \) vector is normal since it is obtained from \( X \) by a linear transformation and it follows from (II), (IV) and (V) that the variance-covariance matrix of this vector is diagonal which determinant equals to \( \sigma^2/n. \)

Hence, its joint density function is
\[
 f(y, \bar{x}_n)(y_2, \ldots, y_n, \bar{x}_n) = \frac{\sqrt{n}}{(2\pi)^{n/2}n^{1/2}} e^{-\frac{1}{2}(\sum_{i=2}^{n} y_i^2 + \frac{1}{\sigma^2}((\bar{x}_n - \mu)^2))}
\]
which can be written as
\[
\frac{1}{(2\pi)^{(n-1)/2}} e^{-\frac{1}{2} \left( \sum_{i=2}^{n} y_i^2 \right)} \frac{\sqrt{n}}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{1}{\sigma^2} (\bar{x} - \mu)^2 \right)} = f_Y(y_2, \ldots, y_n) \cdot f_{\bar{x}_n}(\bar{x}_n)
\]
where the first function is the joint density function of the vector \((Y_2, \ldots, Y_n)\) and the second function is the marginal density function of the variable \(\bar{X}_n\). Therefore, \(Y\) and \(\bar{X}_n\) are independent and so is \(\bar{X}_n\) of any transformation of the \(Y\) vector. Thus, as \(S^2 = \frac{1}{n-1} \sum_{k=2}^{n} Y_k^2\), the independence between the mean and the sample variance is proved.

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